Certifying Concavity and Monotonicity in Games via Sum-of-Squares Hierarchies

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Introduction

Game theory asks how independent decision-makers influence one another. In many modern systems, players choose from a continuum of actions, and we want to know when their interactions settle into predictable, stable outcomes (Nash equilibria). Two structural ideas make this tractable: **concavity** and the stronger notion of **monotonicity**. When these conditions hold, equilibria exist, can even be unique under stronger assumptions, and decentralized learning dynamics can converge to them. Such structure appears in practice, including resource allocation, Cournot competition, and robust power management. The challenge is knowing whether a given game really has these properties. We offer a theoretically grounded certification framework based on **sum-of-squares (SOS) certificates**, checked via well-studied semidefinite programming, which verifies concavity and monotonicity for a given game. These SOS hierarchies yield increasingly tight sufficient conditions, converge in the limit, and remain solvable in polynomial time at each level, providing a systematic route from complex games to provable guarantees.

Contributions

- **SOS hierarchies for verification.**
  We formulate SOS hierarchies to certify concavity and monotonicity in polynomial games with compact, convex, basic semialgebraic action sets.
- **Finite-level certification for generic cases.**
  We show strictly concave or strictly monotone games can be certified at some finite level, and that almost all concave/monotone games admit finite-level certificates.
- **Approximation via ℓ-SOS classes.**
  We define ℓ-SOS-concave and ℓ-SOS-monotone subclasses that globally approximate them, with membership certifiable at level ℓ of the SOS hierarchy.
- **Applications to imperfect-recall EFGs.**
  We demonstrate verification and the nearest ℓ-SOS projections on extensive-form games (EFGs) with imperfect recall.

Polynomial Game over a Basic Semialgebraic Set

Let $\mathscr{G} = \mathscr{G}([\![n]\!], \mathcal{X}, u)$ denote an n-player polynomial game, where $[\![n]\!] \stackrel{\text{def}}{=} \{1, \dots, n\}$, $u = (u_i)_{i\in[\![n]\!]}$ with each $u_i : \mathcal{X} \to \mathbb{R}$ a multivariate polynomial, and

$$\mathcal{X} = \left\{ x \in \mathbb{R}^{m_1\times\cdots\times m_n} \;\middle|\; g_j(x) \geq 0 \;\; \forall j \in [\![m_g]\!],\; h_j(x) = 0 \;\; \forall j \in [\![m_h]\!] \right\}$$

with $g_1, \dots, g_{m_g}, h_1, \dots, h_{m_h}$ polynomials. We assume $\mathcal{X}$ is compact and convex.

Concavity. $\mathscr{G}$ is *concave* iff, for each player $i \in [\![n]\!]$, the Hessian of u_i with respect to x_i is negative semidefinite on $\mathcal{X}$:

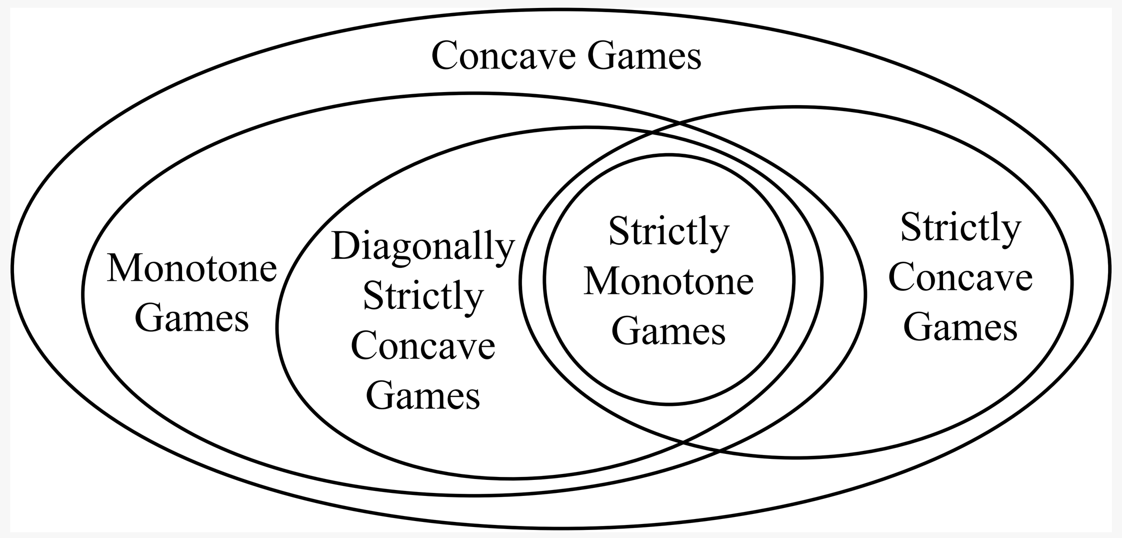
$$\mathbf{H}_{u_i}(x) \stackrel{\text{def}}{=} \nabla^2_{x_i} u_i(x) \preceq 0 \quad \forall x \in \mathcal{X},\ i \in [\![n]\!].$$

Monotonicity. $\mathscr{G}$ is *monotone* iff the symmetrized Jacobian of the concatenated gradient mapping (pseudo-gradient) is negative semidefinite on $\mathcal{X}$:

$$\mathbf{SJ}(x) \stackrel{\text{def}}{=} \frac{1}{2}(\mathbf{J}(x) + \mathbf{J}(x)^{\mathsf{T}}) \preceq 0 \quad \forall x \in \mathcal{X},$$

where $\mathbf{J}(x)$ is the Jacobian of $x \mapsto (\nabla_{x_i} u_i(x))_{i\in[\![n]\!]}$.

Strict forms. $\mathscr{G}$ is *strictly concave* if $\mathbf{H}_{u_i}(x) \prec 0$ on $\mathcal{X}$ for all $i \in [\![n]\!]$, and *strictly monotone* if $\mathbf{SJ}(x) \prec 0$ on $\mathcal{X}$.



Connections and inclusions among the game classes studied in this work.

Hardness of Certifying Concavity/Monotonicity

Theorem 3.1. If for some player i, the payoff u_i is a polynomial of degree at least 3 in x_i, then deciding whether $\mathscr{G}$ is concave or monotone is *strongly NP-hard*.

SOS Hierarchy: Convergence & Genericity

Theorem 3.2. For $\ell \in \mathbb{N}$, define

$$\mathrm{SOS}_\ell \stackrel{\text{def}}{=} \min_{\lambda\in\mathbb{R}} \lambda \quad \text{s.t.} \quad \lambda - y^{\mathsf{T}}\mathbf{SJ}(x)y \in \mathcal{Q}_\ell(\mathcal{X} \times \mathcal{B}),$$

where $\mathcal{B}$ is the unit ball (dimension matches y), and $\mathcal{Q}_\ell(\mathcal{X} \times \mathcal{B})$ is the degree-ℓ truncated quadratic module on (the basic semialgebraic set) $\mathcal{X} \times \mathcal{B}$. Then:

1. **Valid upper bounds.** $\mathrm{SOS}_\ell \geq \max_{x\in\mathcal{X}} \lambda_{\max}(\mathbf{SJ}(x))$, the maximum eigenvalue of $\mathbf{SJ}(x)$.
2. **Tightening bounds.** SOS_ℓ is nonincreasing in ℓ.
3. **Asymptotic optimality.** $\lim_{\ell\to\infty} \mathrm{SOS}_\ell = \max_{x\in\mathcal{X}} \lambda_{\max}(\mathbf{SJ}(x))$.
4. **Optimality in finite time.** If $\mathscr{G}$ is strictly monotone, then $\exists\, \ell < \infty$ with $\mathrm{SOS}_\ell < 0$.

Theorem 3.3. (Genericity). For degree $d \geq 2$, among monotone polynomial games over $\mathcal{X}$, the set that are not strictly monotone has Lebesgue measure zero; hence, for almost all monotone games, monotonicity can be certified at some finite level ℓ.

ℓ-SOS-Monotone Games & Density

Definition. $\mathscr{G}$ is ℓ-SOS-monotone if

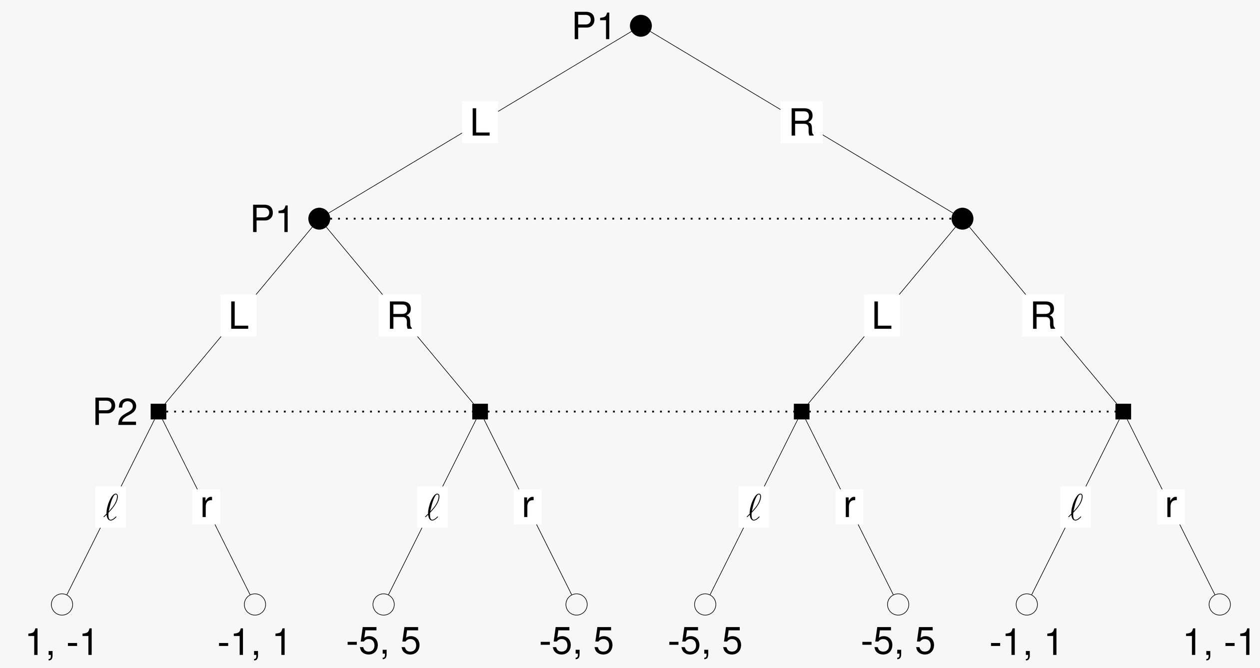
$$-y^{\mathsf{T}}\mathbf{SJ}(x)y \in \mathcal{Q}_\ell(\mathcal{X} \times \mathcal{B}).$$

We call $\mathscr{G}$ *SOS-monotone* if $\mathscr{G}$ is ℓ-SOS-monotone for some finite ℓ.

Theorem 4.3. (Density). For degree $d \geq 2$, SOS-monotone games are dense in monotone games over $\mathcal{X}$. Moreover, for any fixed ℓ and any polynomial game $\mathscr{G}^\star$, the closest ℓ-SOS-monotone game to $\mathscr{G}^\star$ (in SDP-representable norm) can be computed via a single semidefinite program.

Example: Nearest SOS-Monotone EFG

Setup. Consider a two-player zero-sum extensive-form game whose game tree has three levels. At each depth i, all decision nodes belong to a single information set $\mathcal{I}_i$. Player 1 acts at $\mathcal{I}_1$ and $\mathcal{I}_2$; Player 2 acts at $\mathcal{I}_3$. Let x_1 be the probability that Player 1 chooses L at $\mathcal{I}_1$, x_2 the probability that Player 1 chooses L at $\mathcal{I}_2$, and y the probability that Player 2 chooses r at $\mathcal{I}_3$.



A game with no Nash equilibria.

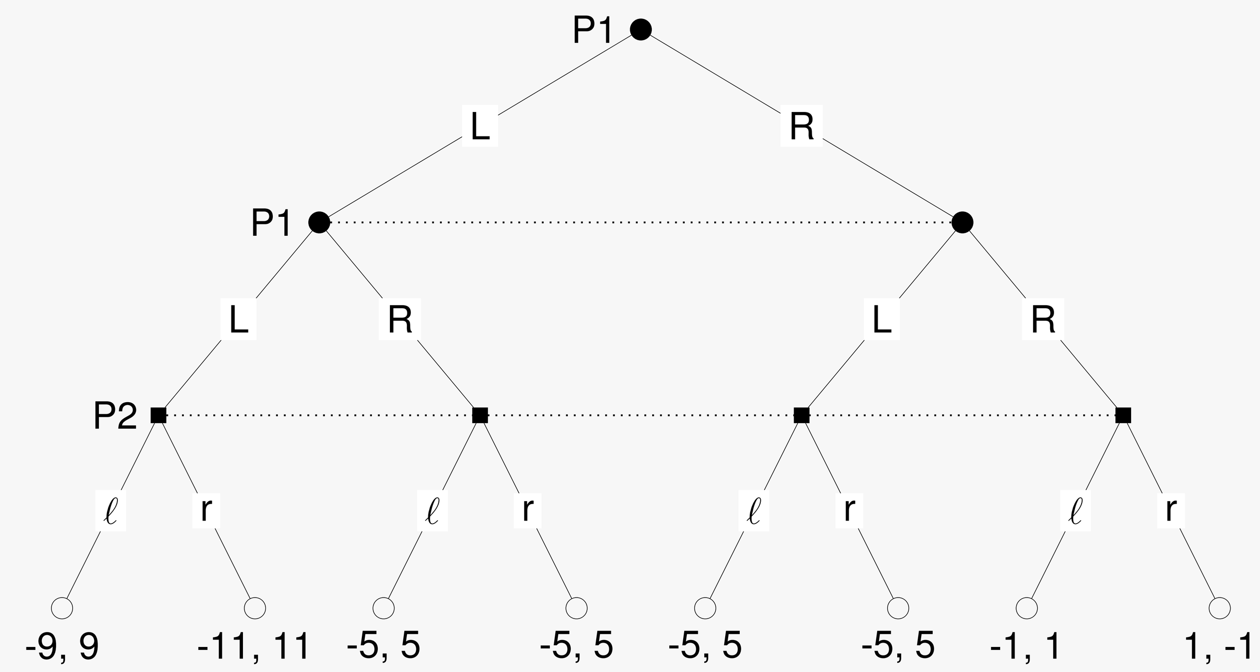
The utilities of this EFG are:

$$u_1(x_1, x_2, y) = 10x_1x_2 + 2x_1y + 2x_2y - 6x_1 - 6x_2 - 2y + 1, \qquad u_2 = -u_1.$$

It has been shown that this EFG has no Nash equilibria.

Nearest SOS-monotone game. The closest zero-sum SOS-monotone game in Euclidean distance is given by the following utilities:

$$u_1(x_1, x_2, y) = 2x_1y + 2x_2y - 6x_1 - 6x_2 - 2y + 1, \qquad u_2 = -u_1.$$



The closest zero-sum SOS-monotone game.

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